

BOCHNER-HARTOGS TYPE EXTENSION THEOREM FOR ROOTS AND LOGARITHMS OF HOLOMORPHIC LINE BUNDLES

S. IVASHKOVICH

ABSTRACT. We prove an extension theorem for roots and logarithms of holomorphic line bundles across strictly pseudoconcave boundaries: they extend in all cases except one, when dimension and Morse index of a critical point is two. In that case we give an explicit description of obstructions to the extension.

CONTENTS

1. Introduction.	1
2. Extension of roots of holomorphic line bundles.	3
3. Extension of roots across the contractible analytic sets.	14
4. Logarithms of line bundles and pseudoconcave domains.	16
5. Appendix. Hartogs domain in a neighborhood of a critical point.	18
References	21

1. INTRODUCTION.

1.1. Bochner-Hartogs type extension. Let X be a complex manifold of dimension $n \geq 2$, U some domain in X and let $\rho : U \rightarrow \mathbb{R}$ be a smooth, strictly plurisubharmonic, Morse function on U , such that for some t_0 the level set $\Sigma_{t_0} := \{\rho = t_0\} \subset U$ is connected and compact. In the sequel, for t close to t_0 , we denote by $U_t^+ := \{x \in U : \rho(x) > t\}$ the upper level set of ρ and by $U_t^- := \{x \in U : \rho(x) < t\}$ - the lower level set. Let a holomorphic line bundle \mathcal{N} on U be given. We say that \mathcal{N} admits a k -th root on $U_{t_0}^+$ if there exists a holomorphic line bundle \mathcal{F} on $U_{t_0}^+$ such that $\mathcal{F}^{\otimes k} \cong \mathcal{N}|_{U_{t_0}^+}$. The main goal of this paper is to understand whether or not a root \mathcal{F} can be extended from $U_{t_0}^+$ to a neighborhood of Σ_{t_0} . It occurs that the answer depends on dimension n of X and on index $\text{ind}\rho|_c$ of a critical point c of ρ (if such exists on Σ_{t_0}). It is clear that without loss of generality we can suppose that the level set Σ_{t_0} is either smooth (i.e., $\text{grad}\rho|_{\Sigma_{t_0}} \neq 0$) or, ρ has exactly one critical point c on Σ_{t_0} of index (necessarily) $0 \leq \text{ind}\rho|_c \leq n = \dim_{\mathbb{C}} X$. Also U should be viewed as a neighborhood of Σ_{t_0} . Our first result can be stated as follows.

Theorem 1. *Suppose that Σ_{t_0} is either smooth*

- i) or, $n \geq 3$ and $\text{ind}\rho|_c$ is arbitrary;*
- ii) or, $n = 2$ and $\text{ind}\rho|_c \neq 2$.*

Then \mathcal{F} extends to a holomorphic line bundle to a neighborhood of Σ_{t_0} and stays there to be a k -th root of \mathcal{N} .

Date: January 19, 2013.

1991 Mathematics Subject Classification. Primary - 32D15, Secondary - 32L10.

Key words and phrases. Holomorphic line bundle, extension of analytic objects.

Now let us turn to the exceptional case $n = 2$ and $\text{ind} = 2$. Consider the following model situation. Denote by $z_j = x_j + iy_j, j = 1, 2$ the coordinates in $\mathbb{C}^2 \cong \mathbb{R}^4$ and let $K^2 = \{|x_j|, |y_j| < 1 : j = 1, 2\}$ be the unit cube in \mathbb{C}^2 . Set $D := K^2 \setminus \{x_1 = x_2 = 0\}$. For every $k \geq 2$ and $1 \leq l < k$ we are going to construct a holomorphic line bundle $\mathcal{F}_k(l)$ on D such that $\mathcal{F}_k(l)^{\otimes k} \cong \mathcal{O}$, i.e., $\mathcal{F}_k(l)$ is a k -th root of the trivial bundle, which doesn't extend to K^2 .

Let $C := \{(x, y) : y_1 = y_2 = 0, x_1^2 + x_2^2 = 1/2\}$ be the generator of $H_1(D, \mathbb{Z})$. Since the totally real plane $\{x_1 = x_2 = 0\}$ is removable for holomorphic functions in D and K^2 is simply connected the exponential $\exp : H^0(D, \mathcal{O}) \rightarrow H^0(D, \mathcal{O}^*)$ is surjective. Moreover, $H^2(D, \mathbb{Z}) = 0$ and therefore the sequence

$$0 \rightarrow H^1(D, \mathbb{Z}) \xrightarrow{j} H^1(D, \mathcal{O}) \xrightarrow{\exp} H^1(D, \mathcal{O}^*) \xrightarrow{c_1} 0 \quad (1.1)$$

is exact. Here $j : H^1(D, \mathbb{Z}) \rightarrow H^1(D, \mathcal{O})$ is the natural imbedding. Denote by A the cohomology class in $H^1(D, \mathbb{Z})$ such that $\langle A, C \rangle = 1$, and by B denote its image $j(A)$ in $H^1(D, \mathcal{O})$. Set further $F_k(l) := \exp(\frac{l}{k}B) := e^{\frac{2\pi il}{k}B} \in H^1(D, \mathcal{O}^*)$ and denote by $\mathcal{F}_k(l)$ the corresponding holomorphic line bundle. Since $(F_k(l))^k = (\exp \circ j)(A) = 0$ we have that $\mathcal{F}_k(l)^{\otimes k} \cong \mathcal{O}$, i.e., $\mathcal{F}_k(l)$ is a k -th root of the trivial bundle on D . At the same time $F_k(l)$ is not zero itself, because $l/k \cdot B$ cannot be an image of anything from $H^1(D, \mathbb{Z})$, and as such the bundle $\mathcal{F}_k(l)$ cannot be extendable to the cube K^2 .

This example shows that in the exceptional case $n = 2, \text{ind} = 2$ the roots of holomorphic line bundles **do not extend** across pseudoconcave boundaries. Indeed the critical case of index two can be easily reduced to the model one, just discussed, via obvious deformations and applying Theorem 1, see Section 2 and Appendix for more details. It turns out and it is our second result, that $\mathcal{F}_k(l)$ exhibit **all** nonextendable k -th roots.

Theorem 2. *In the conditions of Theorem 1 suppose that $n = 2$ and $\text{ind} \rho|_{\mathfrak{c}} = 2$. If \mathcal{F} doesn't extend to a neighborhood of \mathfrak{c} then there exists a neighborhood V of \mathfrak{c} biholomorphic to K^2 such that $\mathcal{F}|_{K^2 \cap U_{t_0}^+}$ extends to D and is isomorphic to some $\mathcal{F}_k(l)$ there.*

This $\mathcal{F}_k(l)$ is uniquely determined by \mathcal{F} , see Remark 2.8.

1.2. Extension of roots across contractible analytic sets. Now let us formulate a general result on extending roots across contractible analytic sets. Recall that a compact analytic set E in a normal complex space X is called **contractible** if there exists a normal complex space Y , a compact analytic set A in Y of codimension at least two and a holomorphic map (a contraction) $C : X \rightarrow Y$ which is a biholomorphism between $X \setminus E$ and $Y \setminus A$. If A is zero dimensional one calls E **exceptional**.

Theorem 3. *Let E be a contractible analytic set in a normal complex space X and let \mathcal{N} be a holomorphic line bundle on X . Suppose \mathcal{N} admits a k -th root \mathcal{F} on $X \setminus E$. Then \mathcal{F} extends to a holomorphic line bundle $\tilde{\mathcal{F}}$ on the whole of X . Moreover, $\tilde{\mathcal{F}}^{\otimes k}$ doesn't depend on k .*

Remark 1. (a) The extension $\tilde{\mathcal{F}}$ of \mathcal{F} can fail to be a k -th root of \mathcal{N} on E . Let X be the blown-up \mathbb{C}^2 in the origin and E be the exceptional curve. Set $\mathcal{N} := [E]$. By the adjunction formula $\mathcal{N}|_E$ is the normal bundle of the imbedding $E \subset X$ i.e., is $\mathcal{O}_E(-1)$. At the same $\mathcal{N}|_{X \setminus E}$ is trivial and as such admits a trivial root $\mathcal{F} = \mathcal{O}$ on $X \setminus E = \mathbb{C}^2 \setminus \{0\}$, and this \mathcal{F} is a root of $\mathcal{N}|_{X \setminus E}$ of any given degree $k \in \mathbb{N}$. But \mathcal{N} doesn't have roots of any degree $k > 1$ on E .

(b) At the same time Theorem 3 means that $\mathcal{N}|_{X \setminus E}$ can be (differently) extended to a line bundle, say $\tilde{\mathcal{N}}$ on X having the extension $\tilde{\mathcal{F}}$ of \mathcal{F} as its k -th root. And, moreover, $\tilde{\mathcal{N}}$ is the same for all k . I.e., we can once forever modify \mathcal{N} on the contractible set and make all roots of \mathcal{N} , which one can take on $X \setminus E$, to extend onto the whole of X as roots of the modified bundle.

(c) Note that according to our definition of a contractible set all compact analytic sets of codimension at least two are contractible. For them, in fact, more is true, see Lemma 3.1: \mathcal{F} extends to a k -th root of \mathcal{N} (and not of some other $\tilde{\mathcal{N}}$).

Remark 2. (a) Among others extension results we prove a Thullen type extension theorem for roots, see Theorem 2.2.

(b) In section 4 we remark that analogous results can be obtained for *logarithms* of holomorphic line bundles.

(c) We end up with an example of non-extendability of roots from pseudoconcave in the sense of Andreotti domains in complex projective plane.

1.3. Acknowledgements. 1. This paper in its final part was accomplished during the authors stay in Math. Department of Nagoya University in September-October 2010. My stay was supported by Grant-in Aid for Scientific Research (A) 22244008 and I'm grateful to Kengo Hirachi and Takeo Ohsawa both for their hospitality and useful discussions. My interest in the questions about extension of roots of holomorphic line bundles was inspired by [Oh].

2. I'm especially grateful to Stefan Nemirovski for explaining to me an example of a non-extendable root from a pseudoconvex in the sense of Andreotti domain in \mathbb{P}^2 . This example is given in section 4.

2. EXTENSION OF ROOTS OF HOLOMORPHIC LINE BUNDLES.

2.1. Extension of analytic objects. Throughout this paper we shall use extension properties of various kind of "analytic objects". These analytic objects have two decisive properties, which we shall formalize in the definition of a *sheaf of analytic objects* below.

Let \mathcal{A}_X be a sheaf of sets on a complex analytic space X .

Definition 2.1. A sheaf \mathcal{A}_X will be called a *sheaf of analytic objects* if it satisfies the following two conditions:

P1) Sections of \mathcal{A}_X obey the uniqueness theorem. I.e., if two of them σ_1, σ_2 are defined on a locally irreducible, connected open subset $U \subset X$ and for some nonempty open $V \subset U$ one has $\sigma_1|_V = \sigma_2|_V$, then $\sigma_1 = \sigma_2$ on U .

P2) The Hartogs type extension lemma is valid for sections of \mathcal{A}_X . I.e., if a section σ of \mathcal{A}_X is given over a Hartogs figure $E_\varepsilon^n := H_\varepsilon^2 \times \Delta^{n-2} \subset X$ then it extends to the section of \mathcal{A}_X over the polydisc Δ^n , $n \geq 2$.

Sections of \mathcal{A}_X will be called *analytic objects* over X .

More precisely in (P2) we mean that this polydisc is imbedded as a open subset $i(\Delta^n)$ into X and σ is initially defined on $i(E_\varepsilon^n)$. Recall that the two dimensional Hartogs figure H_ε^2 in this Definition is

$$H_\varepsilon^2 = \left(\Delta_\varepsilon \times \Delta \right) \cup \left(\Delta \times A_{1-\varepsilon,1} \right), \quad (2.1)$$

where Δ_ε is the disc in \mathbb{C} of radius $\varepsilon > 0$, $A_{1-\varepsilon,1} := \Delta \setminus \overline{\Delta}_{1-\varepsilon}$ is an annulus. Number $n \geq 2$ is the dimension of X in a neighborhood of $i(E_\varepsilon^n)$.

Remark 2.1. (a) The sheaf of holomorphic functions \mathcal{O}_X is a sheaf of analytic objects. The same with meromorphic functions: $\mathcal{M}_X = \text{Mer}(X, \mathbb{P}^1)$. Due to [Iv2] instead of \mathbb{P}^1 one can take any compact Kähler Y .

(b) A smooth holomorphic foliation on X (resp. singular holomorphic foliation) is defined by a holomorphic section (resp. meromorphic section) of the appropriate Grassmann bundle $Gr(TX)$. The products $U \times Gr(T_x X) \equiv Gr(TX)|_U$, where U is a local chart, are Kähler (even projective). Therefore the extension works, i.e., the condition (P2) is satisfied. Involutibility, being an analytic condition, is preserved by extension. Uniqueness condition is also obvious. The extended foliation might become singularities. Therefore the sheaf $\mathcal{F}_X(d)$ of codimension d singular holomorphic foliations on pure dimensional analytic space X is a sheaf of analytic objects.

(c) If a sheaf of objects satisfies the property (P1) then the method of Cartan-Thullen provides the existence of "envelopes", i.e., of maximal domains over a the base space X , where a given family of analytic objects extends. In fact (P1) is not only sufficient but also a necessary condition for that, see [Iv3].

(d) Coherent analytic sheaves/holomorphic bundles are not analytic objects, they do not satisfy neither of conditions (P1) and (P2). This explains the problems with extending them. The same with analytic sets. At the same time analytic subsheaves of coherent analytic sheaves are analytic objects. This was for the first time observed by Siu, see [Si], and morphisms of bundles and sheaves are analytic objects (obvious).

(e) Roots of holomorphic line bundles do satisfy property (P2), see Lemma 2.1 below, but do not satisfy (P1), see example in the Introduction.

(f) Solutions of a general (i.e., non-holomorphic) elliptic system do satisfy (P1) but not (P2).

We are going to state a general Bochner-Hartogs type extension theorem for analytic objects. Recall that a smooth real valued function ρ on a complex manifold U is called *strictly plurisubharmonic* at point x if the Levi form $dd^c \rho(x)$ is positive definite. Here, as well as everywhere, by U_t^+ we denote the superlevel set $\{\rho(x) > t\}$ of ρ , by $\Sigma_t = \{\rho = t\}$ the level set and by U_t^- the lower level set $\{\rho(x) < t\}$.

Theorem 2.1. *Let W and U be domains in a complex manifold X of dimension $n \geq 2$ and let ρ be a strictly plurisubharmonic Morse function on U with $\rho(U) \supset [t_1, t_0]$. Suppose that for every $t \in [t_1, t_0]$*

- i) *the level set Σ_t is connected,*
- ii) *the difference $\Sigma_t \setminus W$ is compact,*
- iii) *and that $U_{t_0}^+ \subset W$.*

Then every analytic object extends from W to $U_{t_1}^+$.

One can think about W as being a complement to a compact in U , for example. The proof of this theorem is standard and goes as follows. Let t be such that our analytic object σ is extended to U_t^+ . All what one needs to do is to place Hartogs figure appropriately near the hypersurface Σ_t and, using (P2) extend σ through them, see Figure 2 for the regular level sets. In the case of critical points one should appropriately put Hartogs

figures near the critical point. In Appendix we give a precise Lemma 5.1 of that kind. The uniqueness condition (P1) will guarantee that all extensions match one with another. And it is at this point where an analogous Bochner-Hartogs type extension theorem fails for the roots of holomorphic line bundles in the critical case - this will be examined later on.

2.2. Generalities on extensions of bundles and sheaves. We refer to the book [GR1] for the generalities on coherent analytic sheaves and to [Si] and [ST] for the specific questions about extensions of sheaves. Here we only briefly give a few, very basic, remarks about extensions of bundles and sheaves. Bundles and their sheaves of sections will be always denoted with the calligraphic letters, like \mathcal{N} or \mathcal{F} . Their total spaces - with the standard letters, like N and F . If a line bundle \mathcal{N} is canonically associated to a divisor D we write $\mathcal{N} = \mathcal{O}(D)$ and $N = [D]$ correspondingly.

1. Let $Y \subset \tilde{Y}$ be domains in a complex manifold (or, a normal complex space) X and let \mathcal{N} be a holomorphic bundle (resp. a coherent analytic sheaf) on Y . One says that \mathcal{N} extends from Y to \tilde{Y} if there exists a holomorphic bundle $\tilde{\mathcal{N}}$ (resp. a coherent analytic sheaf) on \tilde{Y} and an isomorphism $\varphi : \tilde{\mathcal{N}}|_Y \rightarrow \mathcal{N}$ of bundles (resp. of sheaves).

2. If \mathcal{N} is a line bundle then it extends as a bundle if and only if it extends as a coherent analytic sheaf. The way to see this is to pass to the second dual $\tilde{\mathcal{N}}^{**}$ of the extended sheaf, which is reflexive. And then apply Lemma 26 from [Fr] to conclude that $\tilde{\mathcal{N}}^{**}$ is locally free, i.e., is a bundle extension of \mathcal{N} .

3. The feature mentioned in the previous item is specific for line bundles. The rank two subbundle $\mathcal{N} \subset \mathcal{O}^3$ over $\mathbb{C}^3 \setminus \{0\}$ with the stalk $\mathcal{N}_z = \{w : w_1 z_1 + w_2 z_2 + w_3 z_3 = 0\}$ extends to the origin as a coherent sheaf but not as a bundle.

4. Bundles/sheaves do not extend as holomorphic/meromorphic functions, i.e., they are not analytic objects. The following example is very instructive. Take the punctured bidisc $\Delta_*^2 = \{z \in \mathbb{C}^2 : 0 < \max\{|z_1|, |z_2|\} < 1\}$ and cover it by two Stein domains $U_1 = \{z \in \Delta_*^2 : z_1 \neq 0\}$, $U_2 = \{z \in \Delta_*^2 : z_2 \neq 0\}$. Every function of the form $f(z) = e^{\sum_{k,l>0} \frac{a_{kl}}{z_1^k z_2^l}}$ is a transition function of a holomorphic line bundle on Δ_*^2 , which doesn't extend to the origin, provided at least one coefficient $a_{kl} \neq 0$. Indeed, it is easy to see that such a bundle doesn't admit a non identically zero holomorphic section in a punctured neighborhood of the origin. Moreover, all these bundles are pairwise non isomorphic.

In particular, we see that a Hartogs type extension condition (P2) fails for holomorphic bundles. Uniqueness condition (P1) obviously fails to, because all bundles of the same rank are locally isomorphic to the trivial one.

5. At the same time analytic singularities of codimension three are removable for coherent sheaves. The rough reason is that $H^{0,1}(\mathbb{B}_*^3) = 0$, see [Si]. This last remark explains the essential difference between dimension two and dimensions starting from three in the problems addressed in this paper.

6. Let us made the following:

Remark 2.2. (a) Let a holomorphic line bundle \mathcal{F} be defined on a domain V in a complex manifold X and let U be another domain such that $U \cap V$ is nonempty and connected. If $\mathcal{F}|_{U \cap V}$ is trivial then \mathcal{F} extends to $V \cup U$. Indeed, one can use the trivialization, say

$\varphi : \mathcal{F}|_{U \cap V} \rightarrow \mathbb{C} \times (U \cap V)$ as a transition function to glue \mathcal{F} with the trivial bundle $\mathbb{C} \times U$ on U .

(b) At the same time if \mathcal{F} is trivializable on a subdomain $U_1 \subset U \cap V$ then it might be not sufficient to make such a gluing. We shall discuss such situation in the Appendix.

(c) If there are two such domains U_1 and U_2 and, in addition, $U_1 \cap U_2$ and $U_1 \cap U_2 \cap V$ are connected and if \mathcal{F}^i denotes the extensions of \mathcal{F} onto $U_i \cup V$, then we get a transition function $\varphi_{12} : U_1 \cap U_2 \cap V \rightarrow \mathbb{C}^*$. In order that \mathcal{F}^i glue together to a holomorphic line bundle on $U_1 \cup U_2 \cup V$ it is necessary and sufficient that φ_{12} extends to a non-vanishing function on $U_1 \cap U_2$. This last condition is not automatic.

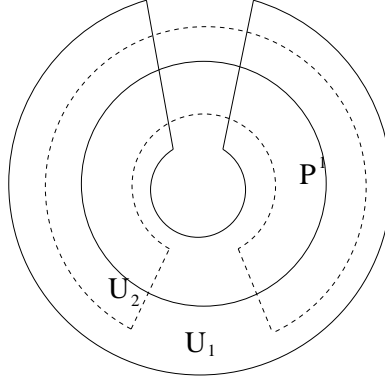


FIGURE 1. The transition function φ_{12} between \mathcal{F}^1 and \mathcal{F}^2 is defined only in $(U_1 \cap U_2) \setminus \mathbb{P}^1$. But it should be defined on $U_1 \cap U_2$ in order for \mathcal{F} to extend.

Example 2.1. Consider the line bundle \mathcal{F} in Δ_*^2 given by the transition function $e^{\frac{1}{z_1 z_2}}$. Blow up the origin and denote by \mathbb{P}^1 the exceptional divisor. Cover it by two bidiscs U_1 and U_2 as on the Figure 1. Then $\mathcal{F}|_{U_j \setminus \mathbb{P}^1}$ is trivial for $j = 1, 2$ (every holomorphic bundle on $\Delta \times \Delta^*$ is trivial) and therefore extends to $\Delta_*^2 \cup U_j$ as a holomorphic line bundle \mathcal{F}^j . But the transition function between \mathcal{F}^1 and \mathcal{F}^2 doesn't extend though $\mathbb{P}^1 \cap U_1 \cap U_2$ and therefore \mathcal{F} doesn't extend onto the blown up bidisc. And it shouldn't, because otherwise its direct image would extend \mathcal{F} downstairs, and this is not the case.

2.3. Hartogs Lemma and Thullen type Extension Theorem for roots. Let us make the first step in the proof of Theorem 1 - a Hartogs type extension lemma for roots of holomorphic line bundles. It occurs to be valid in all dimensions. Recall that the Hartogs figure in \mathbb{C}^n is the following domain

$$H_{\varepsilon, r}^n = \left(\Delta_{\varepsilon}^{n-1} \times \Delta \right) \cup \left(\Delta^{n-1} \times A_{r, 1} \right). \quad (2.2)$$

Here $\Delta := \Delta_1$, and $A_{r, 1} := \Delta \setminus \bar{\Delta}_r$. When one takes $r = 1 - \varepsilon$ one gets the Hartogs figure from the Introduction. But in general the possibility $r = 0$ is not excluded (in that case $A_{0, 1} = \Delta^*$ is the punctured disc), while ε should be strictly positive.

Lemma 2.1. *Let \mathcal{F} be a holomorphic line bundle on the Hartogs figure $H_{\varepsilon, r}^n$, $n \geq 2$, and let $\mathcal{N} = \mathcal{F}^{\otimes k}$ be the tensor k -th power of \mathcal{F} , $k \geq 2$. Suppose that \mathcal{N} extends as a holomorphic line bundle onto the unit polydisc Δ^n . Then \mathcal{F} extends as a holomorphic line bundle onto Δ^n and stays there to be a k -th root of \mathcal{N} .*

Proof. Take the following covering of Δ^n : $U_1 = \Delta_\varepsilon^{n-1} \times \Delta$ and $U_2 = \Delta^{n-1} \times A_{r,1}$. Every holomorphic bundle on $U_j, j = 1, 2$ is trivial and therefore \mathcal{F} is defined in $H_{\varepsilon,r}^n$ by a transition function $f_{12} \in \mathcal{O}^*(U_{12})$, here $U_{12} := U_1 \cap U_2 = \Delta_\varepsilon^{n-1} \times A_{r,1}$. Then f_{12}^k is a transition function of \mathcal{N} . Every holomorphic bundle on Δ^n is trivial and therefore there exist $F_1 \in \mathcal{O}^*(U_1)$ and $F_2 \in \mathcal{O}^*(U_2)$ such that $f_{12}^k = F_1 \cdot F_2$ on U_{12} . U_1 is simply connected and therefore F_1 admits a k -th root f_1 , i.e., such a non-vanishing holomorphic function on U_1 that $f_1^k = F_1$. We have now that $(f_{12} \cdot f_1^{-1})^k = F_2|_{U_{12}}$, i.e., F_2 admits a k -th root on $U_{12} = \Delta_\varepsilon^{n-1} \times A_{r,1}$. But then it obviously admits a k -th root on $\Delta^{n-1} \times A_{r,1} = U_2$. I.e., there exist $f_2 \in \mathcal{O}^*(U_2)$ such that $(f_{12} \cdot f_1^{-1})^k = f_2^k$. All what is left is to multiply f_2 by an appropriate root of 1 to get $f_{12} \cdot f_1^{-1} = f_2$. Therefore \mathcal{F} is trivial and therefore extends to Δ^n .

Let $\varphi : \mathcal{N}|_{H_{\varepsilon,r}^n} \rightarrow \mathcal{F}^{\otimes k}|_{H_{\varepsilon,r}^n}$ be an isomorphism. In global holomorphic frames l of \mathcal{F} and n of \mathcal{N} (over Δ^n) φ is given as $\varphi(n) = h \cdot l^{\otimes k}$, where h is a nonvanishing holomorphic function in $H_{\varepsilon,r}^n$. By the classical Hartogs extension theorem h holomorphically extends to Δ^n and doesn't vanish there. This gives the desired extension of φ and proves that extended \mathcal{F} stays to be a k -th root of \mathcal{N} . □

Remark 2.3. We see that the condition (P2) in the Definition 2.1 of analytic objects is satisfied by roots of holomorphic line bundles. But it is easy to see that (P1) fails. Let X be an Enriques surface, i.e., $\mathcal{K}_X \neq \mathcal{O}_X$ but $\mathcal{K}_X^{\otimes 2} \cong \mathcal{O}_X$. Then for an imbedded bidisc $V \subset X$ we have that $\mathcal{K}_X|_V \cong \mathcal{O}|_V$, but they are different globally. At the same time both these bundles are square roots of the trivial bundle over X .

This was a *global* example. A local one was given in the Introduction.

When $r = 0$ we obtain from Lemma 2.1 a Thullen type extension lemma for the roots of holomorphic line bundles:

Corollary 2.1. *Let \mathcal{F} be a holomorphic line bundle on $(\Delta^{n-1} \times \Delta^*) \cup (\Delta_\varepsilon^{n-1} \times \Delta)$ and let $\mathcal{N} = \mathcal{F}^{\otimes k}$ be the tensor k -th power of \mathcal{F} , $k \geq 2$. Suppose that \mathcal{N} extends as a holomorphic line bundle onto the unit polydisc Δ^n . Then \mathcal{F} extends as a holomorphic line bundle onto Δ^n and stays there to be a k -th root of \mathcal{N} .*

Let us remark that a more general Thullen type extension theorem for roots of holomorphic line bundles is valid in all dimensions (i.e., also in dimension two).

Theorem 2.2. *Let A be an analytic set in a connected complex manifold X and let G be a domain in X , which contains $X \setminus A$ and which intersects every irreducible branch of A of codimension one. Let \mathcal{N} be a holomorphic line bundle on X such that it admits a k -th root \mathcal{F} on G . Then \mathcal{F} extends to a holomorphic line bundle onto the whole of X and stays there to be a k -th root of \mathcal{N} .*

Proof. Let us establish the uniqueness of the extension first. Suppose \mathcal{F} is extended as a k -th root of \mathcal{N} to a bundle \mathcal{F}_1 on an open set $G_1 \supset G$ and, again as a k -th root of \mathcal{N} to a bundle \mathcal{F}_2 on $G_2 \supset G$. Fix an isomorphism $H : \mathcal{F}_1|_G \rightarrow \mathcal{F}_2|_G$. Let us see that H extends to an isomorphism between $\mathcal{F}_1|_{G_1 \cap G_2}$ and $\mathcal{F}_2|_{G_1 \cap G_2}$ and therefore \mathcal{F} extends to $G_1 \cup G_2$ as a k -th root of \mathcal{N} .

Denote by $\text{Reg } A$ the regular part of the codimension one locus of A . Fix a point $q \in \text{Reg } A \cap G$ and take another point $p \in \text{Reg } A \cap (G_1 \cap G_2)$ on the same irreducible component of $\text{Reg } A$ as q . Take now open subsets $q, p \in U_1 \subset \text{Reg } A$ and $U \subset X$ such that

- i) $U_1 = \Delta^{n-1}$ with $q = 0$ and $p = (1/2, 0, \dots, 0)$;
- ii) $U = \Delta^n$ and $U \cap \text{Reg } A = U_1$;
- iii) $\Delta_\varepsilon^{n-1}(q) \times \Delta \subset G$ and $\Delta_\varepsilon^{n-1}(p) \times \Delta \subset G_1 \cap G_2$ for some $\varepsilon > 0$.

Such U_1, U can be found using the Royden's lemma, see [Ro]. Set $V := (U \setminus U_1) \cup (\Delta_\varepsilon^{n-1}(q) \times \Delta) \cup (\Delta_\varepsilon^{n-1}(p) \times \Delta)$.

Claim. \mathcal{F}_i admits a global holomorphic frame over V . Indeed, set $V_1 := (U \setminus U_1) \cup (\Delta_\varepsilon^{n-1}(q) \times \Delta)$ and $V_2 := (U \setminus U_1) \cup (\Delta_\varepsilon^{n-1}(p) \times \Delta)$. By Lemma 2.1 \mathcal{F}_i admits global holomorphic frames g_j over V_j , $j = 1, 2$. We have that $g_1 = hg_2$ on $U \setminus U_1$ for some nonvanishing holomorphic function h on $U \setminus U_1$. Let $\psi_i : \mathcal{F}_i^{\otimes k} \rightarrow \mathcal{N}|_{G_i}$ be an isomorphism. Remark that $\psi_i(g_j^{\otimes k})$ for $j = 1, 2$ are global holomorphic frames of $\mathcal{N}|_{V_j}$ and by the standard Hartogs theorem they extend holomorphically onto U as frames of \mathcal{N} . From this fact we see that h^k and therefore h extend to a nonvanishing holomorphic function on U . I.e., hg_2 is a global holomorphic frame of \mathcal{F}_i over V . The Claim is proved.

Let f_i be global frames of \mathcal{F}_i over V for $i = 1, 2$, constructed in the Claim above. Since $V_1 \subset G$ we see that there exists a nonvanishing holomorphic function h on V_1 such that $H(f_1) = hf_2$. Again by Hartogs theorem h extends to a nonvanishing holomorphic function on U . Therefore we can extend H onto V_2 by setting $H(f_1|_{V_2}) = hf_2|_{V_2}$.

We extended H through every regular point of A . Extension of H through the singular locus of $A \cap G_1 \cap G_2$ goes along the same lines using the stratification of $\text{Sing } A$. Uniqueness of the extension is proved.

Having established the uniqueness of the extension we can finish the proof of our Theorem with the help of Corollary 2.1. Let $\tilde{G} \supset G$ be the maximal domain onto which \mathcal{F} extends as a k -th root of \mathcal{N} . If there exists $p \in \partial\tilde{G} \cap \text{Reg } A$ then we can apply Corollary 2.1 (with $r = 0$) and extend \mathcal{F} to a neighborhood of p . This contradicts to the maximality of \tilde{G} . The removal of $\text{Sing } A$ is analogous.

The condition on the extended bundle to be a k -th root of \mathcal{N} is also preserved by the Thullen type theorem for holomorphic functions.

□

Remark 2.4. We place here a Thullen type extension theorem for roots for the following reasons:

- (a) In the situation when roots of line bundles are not always extendable it is desirable to have complete understanding when they nevertheless do extend.
- (b) In the process of the proof it was very clear how the uniqueness property plays a crucial role in extension.
- (c) The proof of the extendability through subcritical points in dimension two and through critical one in dimension ≥ 3 is analogous to that given for Thullen case.

2.4. Extension of roots across regular level sets. We shall make a first step in the proof of Bochner-Hartogs type Extensions Theorems 1 and 2 for roots of holomorphic line bundles. For the convenience of the future references we shall state Theorems 1 and 2 in a slightly more general, but obviously equivalent form. As in Theorem 2.1 we consider two domains W and U in a complex manifold X of dimension $n \geq 2$ and let ρ be a strictly

plurisubharmonic Morse function on U with $\rho(U) \supset [t_1, t_0]$. We shall suppose that for all $t \in [t_1, t_0]$ the level set Σ_t is connected, $\Sigma_t \setminus W$ is compact and that $U_{t_0}^+ \subset W$.

Theorem 2.3. *Let \mathcal{N} be a holomorphic line bundle on X and suppose that it admits a k -th root \mathcal{F} on W for some $k \geq 2$. Then:*

i) \mathcal{F} extends to a k -th root of \mathcal{N} on $U_{t_1}^+$ provided $n = \dim U \geq 3$ or, $n = 2$ and neither of Σ_t -s has critical points of index 2.

ii) If $n = 2$, Σ_{t^*} for some $t^* < t_0$ contains some critical point \mathbf{c} of index 2, and \mathcal{F} is extended to $U_{t^*}^+$ as a k -th root of \mathcal{N} then either \mathcal{F} extends to a neighborhood of \mathbf{c} or, there exists a neighborhood V of \mathbf{c} biholomorphic to K^2 such that $\mathcal{F}|_{K^2 \cap U_{t^*}^+}$ extends to D and is isomorphic there to some uniquely defined $\mathcal{F}_k(l)$.

Proof. By the assumption \mathcal{F} exists as a k -th root of $\mathcal{N}|_{U_{t_0}^+}$ on $U_{t_0}^+$. Let $\psi_{t_0} : \mathcal{N}|_{U_{t_0}^+} \rightarrow \mathcal{F}^{\otimes k}$ be a corresponding isomorphism. Denote by t^* the infimum of such $t \leq t_0$ that \mathcal{F} extends up to t as a k -th root \mathcal{F}_t of $\mathcal{N}|_{U_t^+}$. More precisely, by saying that \mathcal{F} extends up to t as a k -th root \mathcal{F}_t of \mathcal{N} , we mean the following:

a) Holomorphic line bundles \mathcal{F}_{t_1} on $U_{t_1}^+$ are defined for all $t_1 \geq t$ together with isomorphisms $\psi_{t_1} : \mathcal{N}|_{U_{t_1}^+} \rightarrow \mathcal{F}_{t_1}^{\otimes k}$.

b) For all pairs $t_1 \geq t_2 \geq t$ the bundle \mathcal{F}_{t_2} is an extension of \mathcal{F}_{t_1} , i.e., an isomorphism $\varphi_{t_1 t_2} : \mathcal{F}_{t_2}|_{U_{t_1}^+} \rightarrow \mathcal{F}_{t_1}$ is given, and these isomorphisms satisfy:

b)₁ $\varphi_{t_1 t_1} = \text{Id}$;

b)₂ $\varphi_{t_1 t_2} \circ \varphi_{t_2 t_3} = \varphi_{t_1 t_3}$ for $t_1 \geq t_2 \geq t_3$.

c) Isomorphisms ψ_{t_1} and $\varphi_{t_1 t_2}$ are natural in the sense that for every pair $t_1 \geq t_2 \geq t$ the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{N}|_{U_{t_2}^+} & \xrightarrow{r_{t_1 t_2}} & \mathcal{N}|_{U_{t_1}^+} \\ \downarrow \psi_{t_2} & & \downarrow \psi_{t_1} \\ \mathcal{F}_{t_2}^{\otimes k} & \xrightarrow{\varphi_{t_1 t_2}^{\otimes k}} & \mathcal{F}_{t_1}^{\otimes k} \end{array} \quad (2.3)$$

Here $r_{t_1 t_2} : \mathcal{N}|_{U_{t_2}^+} \rightarrow \mathcal{N}|_{U_{t_1}^+}$ is the natural restriction operator of the globally existing bundle \mathcal{N} .

Step 1. The infimum t^* is achieved. If t^* is the infimum then for every $t > t^*$ the bundle \mathcal{F} extends onto U_t^+ as a k -th root \mathcal{F}_t of \mathcal{N} . We define the presheaf \mathcal{F}_{t^*} on $U_{t^*}^+$ as the projective limit of \mathcal{F}_t -s for $t > t^*$:

$$\mathcal{F}_{t^*} := \varprojlim \mathcal{F}_t. \quad (2.4)$$

A section σ of \mathcal{F}_{t^*} over an open set $V \subset U_{t^*}^+$ is a product $\prod_{t > t^*} \sigma_t$ of sections σ_t of \mathcal{F}_t over $V \cap U_t^+$ such that $\varphi_{t_1 t_2}(\sigma_{t_2}) = \sigma_{t_1}$ for every pair $t_1 \geq t_2 > t^*$. Restriction map for $W \subset V$ in \mathcal{F}_{t^*} is $\prod_{t > t^*} \sigma_t \rightarrow \prod_{t > t^*} \sigma_t|_{W \cap U_t^+}$, which is correctly defined.

It is easy to see that the presheaf, so defined, is actually a sheaf and that this sheaf is locally free of rank one. Isomorphisms $\varphi_{tt^*} : \mathcal{F}_{t^*}|_{X_t} \rightarrow \mathcal{F}_t$ for $t > t^*$ are naturally defined as $\varphi_{tt^*}(\prod_{t > t^*} \sigma_t) = \sigma_t$. For $t = t^*$ we set $\varphi_{t^* t^*} = \text{Id}$.

Take some $t_1 > t^*$. It is not difficult now to see that $\psi_{t_1} : \mathcal{N}|_{U_{t_1}^+} \rightarrow \mathcal{F}_{t_1}^{\otimes k}$ extends to an isomorphism $\psi_{t^*} : \mathcal{N}|_{U_{t^*}^+} \rightarrow \mathcal{F}_{t^*}^{\otimes k}$ and the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{N}|_{U_{t^*}^+} & \xrightarrow{r_{t_1 t^*}} & \mathcal{N}|_{U_{t_1}^+} \\
\downarrow \psi_{t^*} & & \downarrow \psi_{t_1} \\
\mathcal{F}_{t^*}^{\otimes k} & \xrightarrow{\varphi_{t_1 t^*}^{\otimes k}} & \mathcal{F}_{t_1}^{\otimes k}.
\end{array} \tag{2.5}$$

The proof goes by continuity: for, if ψ_{t_1} is already extended to ψ_t , then take the composition $(\varphi_{tt^*}^{-1})^{\otimes k} \circ \psi_t : \mathcal{N}|_{X_t} \rightarrow \mathcal{F}_{t^*}^{\otimes k}|_{X_t}$. It is an analytic morphism of sheaves and in local charts it is given by holomorphic functions. Therefore it can be extended through a pseudoconcave boundary.

Remark 2.5. On this step we do not need to suppose that Σ_{t^*} is regular.

In the remaining steps we shall bypass t^* . In the process of the proof we shall distinguish the case when Σ_{t^*} is regular from the case when it is a critical level of some strictly plurisubharmonic Morse function, say ρ .

Step 2. Passing through a regular value. Let t^* be a regular value of ρ and suppose that \mathcal{F}_{t^*} is an extension of \mathcal{F} to $U_{t^*}^+$, which stays to be a k -th root of \mathcal{N} there. Take a point $z_0 \in \Sigma_{t^*}$. Then in an appropriate coordinates near z_0 one can find a polydisc Δ^n such that $\Delta^n \cap U_{t^*}^+$ is connected, $z_0 = (\frac{1}{2}, 0, \dots, 0)$ and such that $H_\varepsilon^n \subset U_{t^*}^+$ for some $\varepsilon > 0$. By Lemma 2.1 the restriction $\mathcal{F}_{t^*}|_{H_\varepsilon^n}$ is trivial; let $\varphi : \mathcal{F}_{t^*}|_{H_\varepsilon^n} \rightarrow \mathbb{C} \times H_\varepsilon^n$ be a trivialization. Extend it by the usual Hartogs theorem to an isomorphism $\varphi : \mathcal{F}_{t^*}|_{\Delta^n \cap U_{t^*}^+} \rightarrow \mathbb{C} \times (\Delta^n \cap U_{t^*}^+)$. Then it defines (as a transition function) a bundle $\tilde{\mathcal{F}}$ over $U_{t^*}^+ \cup \Delta^n$ - an extension of \mathcal{F}_{t^*} . Furthermore, by assumption there exists an isomorphism $\psi : \mathcal{F}_{t^*}^{\otimes k}|_{U_{t^*}^+} \rightarrow \mathcal{N}|_{U_{t^*}^+}$. The isomorphism $\psi \circ (\varphi^{-1})^{\otimes k} : \tilde{\mathcal{F}}|_{\Delta^n \cap U_{t^*}^+}^{\otimes k} \rightarrow \mathcal{N}|_{\Delta^n \cap U_{t^*}^+}$ obviously extends to Δ^n (again by the classical Hartogs theorem) and this gives an extension of ψ onto $U_{t^*}^+ \cup \Delta^n$. Now we can cover Σ_{t^*} by a finite number of such coordinate neighborhoods U_j that:

- i) Each U_j is biholomorphic to a polydisc Δ^n and $U_j \cap U_{t^*}^+$ are connected;
- ii) the associated Hartogs figures H_ε^n are contained in $U_j \cap U_{t^*}^+$;
- iii) the double $U_{ij} := U_i \cap U_j$ and triple $U_{ijk} := U_i \cap U_j \cap U_k$ intersections are connected, simply connected and moreover, $U_{ij} \cap U_{t^*}^+$ and $U_{ijk} \cap U_{t^*}^+$ are connected and simply connected as well.

Extend \mathcal{F}_{t^*} through all of them using Lemma 2.1, i.e., get line bundles $\mathcal{F}_{t^*}^j$ over $V_j := U_{t^*}^+ \cup U_j$ - extensions of \mathcal{F}_{t^*} . We need to prove that these extensions match together, i.e., that transition functions extend from $U_i \cap U_j \cap U_{t^*}^+$ to $U_i \cap U_j$ for all $U_i \cap U_j \neq \emptyset$. Denote by $\varphi_j : \mathcal{F}_{t^*}^j|_{U_{t^*}^+} \rightarrow \mathcal{F}_{t^*}$ the corresponding isomorphisms. Then we get the transition functions $\varphi_{ij} = \varphi_i^{-1} \circ \varphi_j : \mathcal{F}_{t^*}^j|_{U_i \cap U_j \cap U_{t^*}^+} \rightarrow \mathcal{F}_{t^*}^i|_{U_i \cap U_j \cap U_{t^*}^+}$. As it was said already, we need to extend φ_{ij} onto $U_i \cap U_j$ (this is not automatic, see Fig. 2).

Denote by $\psi_j : \mathcal{N}|_{V_j} \rightarrow (\mathcal{F}_{t^*}^j)^{\otimes k}$ the isomorphisms, obtained in Lemma 2.1 as extensions of the isomorphism $\psi_{t^*} : \mathcal{N}|_{U_{t^*}^+} \rightarrow \mathcal{F}_{t^*}^{\otimes k}$. From the diagram

$$\begin{array}{ccc}
\mathcal{N}|_{U_i \cap U_j \cap U_{t^*}^+} & \xrightarrow{\text{tr}} & \mathcal{N}|_{U_i \cap U_j \cap U_{t^*}^+} \\
\downarrow \psi_j & & \downarrow \psi_i \\
(\mathcal{F}_{t^*}^j|_{U_i \cap U_j \cap U_{t^*}^+})^{\otimes k} & \xrightarrow{\varphi_{ij}^{\otimes k}} & (\mathcal{F}_{t^*}^i|_{U_i \cap U_j \cap U_{t^*}^+})^{\otimes k}
\end{array} \tag{2.6}$$

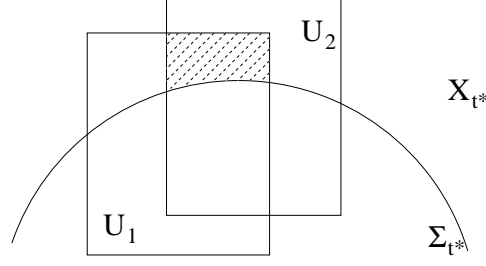


FIGURE 2. The transition function φ_{12} between $\mathcal{F}_{t^*}^1$ and $\mathcal{F}_{t^*}^2$ is defined only in $U_1 \cap U_2 \cap U_{t^*}^+$ - dashed zone on the picture. But it should be defined on $U_1 \cap U_2$. Since the envelope of holomorphy of $U_1 \cap U_2 \cap U_{t^*}^+$ is much smaller than $U_1 \cap U_2$ the extension of φ_{12} to $U_1 \cap U_2$ is not automatic, i.e., cannot be achieved via the classical Hartogs theorem.

we see that $\varphi_{ij}^{\otimes k} = \psi_i \circ \psi_j^{-1}$ on $U_i \cap U_j \cap U_{t^*}^+$. But ψ_j (resp. ψ_i) is defined over U_j (resp. U_i) and tr is a transition map of a globally existing bundle \mathcal{N} , therefore $\psi_i \circ \text{tr} \circ \psi_j^{-1}$ is defined over $U_i \cap U_j$ extending $\varphi_{ij}^{\otimes k}$. But then φ_{ij} also extends onto $U_i \cap U_j$ as a k -th root on an extendable non-vanishing function on a simply connected domain.

It is easy to see that the cocycle condition for the extended transition maps will be preserved. Indeed, it is satisfied on $U_{ijk} \cap U_{t^*}^+$, so it will be satisfied on U_{ijk} to.

2.5. Line bundles with vanishing Chern class. Before considering the case of critical points let us make few preparatory remarks. Let X be a connected (not necessarily compact) complex manifold. Then we have the following exact sequence:

$$0 \rightarrow \mathbb{Z} \rightarrow H^0(X, \mathcal{O}) \xrightarrow{\exp} H^0(X, \mathcal{O}^*) \rightarrow H^1(X, \mathbb{Z}) \rightarrow H^{0,1}(X) \rightarrow \text{Pic}^0(X) \rightarrow 0, \quad (2.7)$$

where $\text{Pic}^0(X)$ is the group of topologically trivial holomorphic line bundles on X . $\text{Pic}^0(X)$ appears here as the kernel of the map c_1 from the group $H^1(X, \mathcal{O}^*)$ of all holomorphic line bundles on X to the Néron-Severi group $NS(X)$. I.e., $\text{Pic}^0(X)$ is exactly the group of holomorphic line bundles on X with vanishing first Chern class. The arrow $H^{0,1}(X) \rightarrow \text{Pic}^0(X)$ is the composition of the Dolbeault isomorphism $D : H^{0,1}(X) \rightarrow H^1(X, \mathcal{O})$ and exponential map. If the map

$$H^0(X, \mathcal{O}) \xrightarrow{\exp} H^0(X, \mathcal{O}^*)$$

is surjective then (2.7) writes as

$$0 \rightarrow H^1(X, \mathbb{Z}) \rightarrow H^{0,1}(X) \rightarrow \text{Pic}^0(X) \rightarrow 0, \quad (2.8)$$

which means that $\text{Pic}^0(X) \cong H^{0,1}(X)/H^1(X, \mathbb{Z})$. This is the case for example if all holomorphic functions on X are constant or, all holomorphic functions from X extend to some simply connected $\tilde{X} \supset X$. The pair $K^2 \supset D := K^2 \setminus \{x_1 = x_2 = 0\}$ from the example in the Introduction is the case we are particularly going to exploit.

2.6. Extension through subcritical points. Now we shall consider the case with critical points. Let us make first of all some standard observations. We follow computations from [Iv1], though other sources also might be used at this place. Let t^* be a critical value of our strictly plurisubharmonic Morse function ρ and let z_0 be a critical point of

ρ on $\Sigma_{t^*} := \{\rho(x) = t^*\}$. In an appropriate coordinates near $z_0 = 0$ (and make $t^* = 0$ for convenience) the function ρ has the form

$$\rho(z) = \sum_{j=1}^n a_j z_j^2 + \sum_{j=1}^n a_j \bar{z}_j^2 + \sum_{j=1}^n |z_j|^2 + O(\|z\|^3) \quad (2.9)$$

with some real non-negative coefficients a_1, \dots, a_n . In coordinates $z_j = x_j + iy_j$ we rewrite (2.9) as follows

$$\begin{aligned} \rho(z) &= 2 \sum_{j=1}^n a_j (x_j^2 - y_j^2) + \sum_{j=1}^n (x_j^2 + y_j^2) + O(\|z\|^3) = \\ &= \sum_{j=0}^p \left[(1 + 2a_j) x_j^2 + (1 - 2a_j) y_j^2 \right] + \sum_{j=p+1}^n \left[(1 + 2a_j) x_j^2 + (1 - 2a_j) y_j^2 \right] + O(\|z\|^3), \end{aligned} \quad (2.10)$$

where the index p has the following significance: $a_j > \frac{1}{2}$ for $j \leq p$ and $0 \leq a_j < \frac{1}{2}$ for $j \geq p+1$. Number p is in fact the index of the Morse function ρ at 0. Case $a_j = \frac{1}{2}$ is excluded by the choice of ρ to be Morse, case $p = 0$ is not excluded. Taking $\delta > 0$ and $\|z\|$ sufficiently small we can write

$$\begin{aligned} \rho(z) &\geq \sum_{j=0}^p [(2a_j + 1)x_j^2 - (2a_j - 1)y_j^2] + \delta \cdot \sum_{j=p+1}^n |z_j|^2 + O(\|z\|^3) \geq \\ &\geq a^2 \sum_{j=0}^p x_j^2 - b^2 \sum_{j=0}^p y_j^2 + \delta^2 \cdot \sum_{j=p+1}^n |z_j|^2 := \rho_1(z) \end{aligned} \quad (2.11)$$

with some $a > b$. From (2.11) we see that $Y^+ := \{z \in \Delta^n : \rho_1(z) > 0\} \subset U_{t^*}^+$ near the origin. Therefore all that we need is to extend the root \mathcal{F} from Y^+ to $Y^+ \cup \{\text{a neighborhood of the origin}\}$. Denote the coordinates (z_1, \dots, z_p) as z with $z = x + iy$ in vector notations, and (z_{p+1}, \dots, z_n) by w . Set $A = \frac{a}{b} > 1$, $K_1 = \{x \in \mathbb{R}^p : \|x\| < 1\}$, $K_A = \{y \in \mathbb{R}^p : \|y\| < A\}$ and $K^2 = K_1 \times K_A \times \Delta^{n-p}$. Appropriately deforming $\Sigma_{t^*}^+$ and rescaling coordinates near the origin we can suppose that

$$Y^+ := \{(z, w) \in K^2 : \rho_1(z) = a^2 \|x\|^2 - b^2 \|y\|^2 + \delta^2 \|w\|^2 > 0\}. \quad (2.12)$$

Sep 3. Subcritical case, i.e., $0 \leq p \leq n-1$. The case $p = 0$ is obviously served by Hartogs type Lemma 2.1.

Lemma 2.2. *In the case $1 \leq p \leq n-1$ domain Y^+ is simply connected.*

Proof. This is immediate. Indeed, from (2.12) we see that for z such $a^2 \|x\|^2 - b^2 \|y\|^2 > 0$ the point w , with $(z, w) \in Y^+$, belongs to an appropriate $(n-p)$ -disc; if $a^2 \|x\|^2 - b^2 \|y\|^2 = 0$ then w belongs to a punctured $(n-p)$ -disc; and, finally, if $a^2 \|x\|^2 - b^2 \|y\|^2 < 0$ then w lies in an appropriate ring domain of the type $A_{r_1, r_2}^{n-p} := \Delta_{r_1}^{n-p} \setminus \bar{\Delta}_{r_2}^{n-p}$. It is important here that $n-p > 0$. Now the w -coordinate of every loop in Y^+ can be moved to this ring domain and then z -coordinate retracted to a point. One then moves this point to a point inside $\{a^2 \|x\|^2 - b^2 \|y\|^2 < 0\}$ and finally contracts the loop. \square

Since \mathcal{N} is extendable from Y^+ to K^2 we have that $c_1(\mathcal{N}|_{K^2}) = 0$ and therefore $c_1(\mathcal{F}|_{Y^+}) = 0$. As the result $\mathcal{F}|_{Y^+}$ belongs to $H^{0,1}(Y^+)$. Let ω be a $(0,1)$ -form in Y^+

representing $\mathcal{F}|_{Y^+}$ and let τ be a $(0,1)$ -form in K^2 representing $\mathcal{N}|_{K^2}$. Then, since $\mathcal{F}^{\otimes k}$ extends to K^2 and is equal to \mathcal{N} there, and because $H^1(Y^+, \mathbb{Z}) = 0$ one has

$$\tau|_{Y^+} = k\omega + \bar{\partial}f$$

for some smooth function f in Y^+ . But $\mathcal{N}|_{K^2}$ is trivial and therefore $\tau = \bar{\partial}g$ for some smooth function g on K^2 . Therefore

$$\omega = 1/k(\bar{\partial}g - \bar{\partial}f),$$

which means that \mathcal{F} is trivial on Y^+ to. Now one can extend \mathcal{F} to $K^2 \cup U_{t^*}^+$ as in Remark 2.2. The case is proved.

2.7. Critical case. Now we want to bypass critical points of maximal index $p = n$. In that case we have that $K^2 = K_1 \times K_A$ is an open brick in \mathbb{C}^n with $K_1 = \{x \in \mathbb{R}^n : \|x\| < 1\}$, $K_A = \{y \in \mathbb{R}^n : \|y\| < A\}$ and

$$Y^+ = \{z \in K^2 : a^2 \|x\|^2 > b^2 \|y\|^2\} = \{z \in K^2 : A^2 \|x\|^2 > \|y\|^2\}. \quad (2.13)$$

Consider one more strictly plurisubharmonic function

$$\rho_2(z) = A^2 \|x\|^2 - \|y\|^2. \quad (2.14)$$

Step 4. Critical case in dimension ≥ 3 . In that case Y^+ is again obviously simply connected and the same proof as in subcritical case gives an extension of \mathcal{F} to a neighborhood of the critical point.

Theorem 1 together with the part (i) of Theorem 2.3 are proved.

Now let us make a preparatory remark for the proof of Theorem 2. Denote by $Y_\tau := \{z \in K^2 : \rho_2(z) > \tau\}$ - the superlevel set of ρ_2 and by $S_\tau := \{z \in K : \rho_2(z) = \tau\}$ denote the level set. Then $Y^+ = Y_0$ in our new notations.

Remark 2.6. Using the fact that $A > 1$ one can deform S_τ to hypersurfaces \tilde{S}_τ in such a way that they will stay smooth and strictly pseudoconvex for all $\tau > 0$ and moreover, they will exhaust the following domain $D_{\delta,\varepsilon}$ for an appropriate $0 < \varepsilon \ll \delta \ll 1$. Namely, set $K_\delta := \{(x_1, x_2) : -\delta < x_1, x_2 < \delta\}$ and $K_\varepsilon := \{(y_1, y_2) : -\varepsilon < y_1, y_2 < \varepsilon\}$. Then set also $K_{\delta,\varepsilon}^2 := K_\delta \times K_\varepsilon$, and define $D_{\delta,\varepsilon} := K_{\delta,\varepsilon}^2 \setminus \{x_1 = x_2 = 0\}$.

2.8. Case of index and dimension two. Let us start this Subsection with a remark about torsion bundles. Denote by $\mathbb{Z}_k = \{e^{\frac{2\pi il}{k}} : 0 \leq l \leq k-1\}$ the group of k -th roots of 1.

Remark 2.7. Let \mathcal{E} be a torsion bundle of order k on a manifold V . Let us see that it can be defined by a locally constant cocycle $\varepsilon_{jl} \in \mathbb{Z}_k$ in an appropriate fine covering $\{U_j\}$ of V . Indeed, let $\{U_j\}$ be a Stein covering with simply connected U_j and let e_{jl} be a defining cocycle for \mathcal{E} in this covering. Since $\mathcal{E}^{\otimes k}$ is trivial there exist holomorphic nonvanishing g_j in U_j such that $g_j e_{jl}^k g_l^{-1} = 1$. Take k -th roots \tilde{g}_j of g_j on U_j arbitrarily. Then $\tilde{e}_{jl} := \tilde{g}_j e_{jl} \tilde{g}_l^{-1}$ will be still a defining cocycle for \mathcal{E} . But now $\tilde{e}_{jl}^k = 1$, i.e., $\tilde{e}_{jl} \in \mathbb{Z}_k$.

We are prepared to give the proof of Theorem 2. Let $\mathbf{c} \in \Sigma_{t_0}$ be a critical point in question. Choosing coordinates around $\mathbf{c} = 0$ appropriately and deforming Σ_t for $t > t_0$ (equivalently Y_τ for $\tau > 0$) near \mathbf{c} , as it was explained in the previous subsection, we can achieve, using Theorem 1 the extendability of \mathcal{F} onto $D_{\delta,\varepsilon} := K_\delta \times K_\varepsilon \setminus \{x_1 = x_2 = 0\}$ in these coordinates.

Cover $D_{\delta,\varepsilon}$ by four Stein domains $D_1 := \{x_1 > 0\} \cap D_{\delta,\varepsilon}$, $D_2 := \{x_2 > 0\} \cap D_{\delta,\varepsilon}$, $D_3 := \{x_1 < 0\} \cap D_{\delta,\varepsilon}$, $D_4 := \{x_2 < 0\} \cap D_{\delta,\varepsilon}$. For naturals $1 \leq l < k$ a holomorphic line bundle $\mathcal{F}_k(l)$ on $D_{\delta,\varepsilon}$ from Introduction is defined by the following cocycle $\{f_{ij}\} \in H^1(\mathcal{O}^*, \{D_i\})$:

$$f_{ij} = \begin{cases} e^{\frac{2\pi i l}{k}} & \text{if } i = 1, j = 2; \\ 1 & \text{otherwise.} \end{cases} \quad (2.15)$$

Lemma 2.3. *Let \mathcal{F} be a torsion bundle on $D_{\delta,\varepsilon}$ of order $k \geq 2$. Then \mathcal{F} is isomorphic to one of $\mathcal{F}_k(l)$.*

Proof. Since $H^2(D_{\delta,\varepsilon}, \mathbb{Z}) = 0$ we have that $c_1(\mathcal{F}) = 0$ and therefore \mathcal{F} belongs to $H^{0,1}(D_{\delta,\varepsilon})/H^1(D_{\delta,\varepsilon}, \mathbb{Z})$, where $H^1(D_{\delta,\varepsilon}, \mathbb{Z})$ is imbedded to $H^{0,1}(D_{\delta,\varepsilon})$ as it was explained in Subsection 2.5. Let ω be a $\bar{\partial}$ -closed $(0,1)$ -form representing \mathcal{F} . Since $\mathcal{F}^{\otimes k}$ is extendable onto $K_{\delta,\varepsilon}^2$ we see that there exists a $\bar{\partial}$ -closed $(0,1)$ -form τ on $K_{\delta,\varepsilon}^2$ such that

$$\tau|_{D_{\delta,\varepsilon}} = k \cdot \omega + \bar{\partial}f + e, \quad (2.16)$$

where f is a smooth function on $D_{\delta,\varepsilon}$ and $e \in H^1(D_{\delta,\varepsilon}, \mathbb{Z})$. Dividing (2.16) by k we obtain

$$1/k \cdot \tau|_{D_{\delta,\varepsilon}} = \omega + \bar{\partial}(1/k \cdot f) + 1/k \cdot e, \quad (2.17)$$

i.e., we see that $\mathcal{F} \otimes \mathcal{E}$ extends to $K_{\delta,\varepsilon}^2$, where \mathcal{E} is the torsion bundle which corresponds to the cohomology class $1/k \cdot e \in H^{0,1}(D_{\delta,\varepsilon})$. I.e., $\mathcal{F} \otimes \mathcal{E}$ is trivial and, therefore, \mathcal{F} is isomorphic to the torsion bundle \mathcal{E}^{-1} . But \mathcal{E}^{-1} can be only one of $\mathcal{F}_k(l)$ by construction. \square

To finish the proof of Theorem 2 all what is left is to remark that the restriction of \mathcal{F} to $D_{\delta,\varepsilon}$ is a torsion bundle, because $\mathcal{F}^{\otimes k}$ is isomorphic to \mathcal{N} and the last is trivial on $K_{\delta,\varepsilon}^2$. If $\mathcal{F}|_{D_{\delta,\varepsilon}}$ is trivial itself then \mathcal{F} extends to a neighborhood of \mathbf{c} . Otherwise it is isomorphic to one of $\mathcal{F}_k(l)$. Theorem 2 is proved. \square

Remark 2.8. We proved that a k -th root \mathcal{F} of a globally defined holomorphic line bundle near a pseudoconcave boundary point of index 2 in dimension 2 possesses a numerical invariant $1 \leq l \leq k-1$. This l is correctly defined. Indeed if \mathcal{F} is defined say on Y_τ for some $\tau > 0$ then its extension to Y_0 is determined uniquely, because morphisms of holomorphic bundles are analytic objects.

3. EXTENSION OF ROOTS ACROSS THE CONTRACTIBLE ANALYTIC SETS.

In this section we are going to prove Theorem 3 from the Introduction and derive corollaries from it.

3.1. Extension of analytic coverings. For the proof of Theorem 3 we shall need the removability of codimension two singularities of analytic coverings. The needed statement is implicitly contained in [NS] and [De], but neither of this papers doesn't contain an explicit formulation. Therefore we shall sketch the proof of the forthcoming Theorem 3.1, refereing step by step to the quoted texts (more to [De]) and keeping notations close to that ones in [De]. Recall that an analytic covering is a proper, surjective holomorphic map $\mathbf{c} : \tilde{X} \rightarrow X$ between normal complex spaces of equal dimensions.

Theorem 3.1. *Let X be a normal complex space and A is an analytic subset of X of codimension at least two. Then every analytic covering \tilde{X} over $X \setminus A$ extends to an analytic covering over the whole of X .*

Remark 3.1. Let us remark that analytic coverings are not analytic objects. They don't obey neither uniqueness property (P1) nor the Hartogs type extension property (P2).

Proof. First of all let us make a few reductions. The problem is local and therefore, since every analytic space can be realized as an analytic covering of a domain in a complex linear space, one reduces the general statement to the case when X is a pseudoconvex domain in \mathbb{C}^n . After that let us remark that the branch locus Y of the covering, being a hypersurface, extends through the codimension two set A by Remmert-Stein theorem. Denote this extended hypersurface as \tilde{Y} . If A contains points from $X \setminus \tilde{Y}$ then, since the complement to \tilde{Y} in the smooth locus of X is locally simply connected the extendability of the covering over such points follows. Therefore we can suppose that $A \subset \tilde{Y}$ in the sequel.

Let us remark furthermore that all we need is to separate the sheets of the covering $\tilde{X} \rightarrow X \setminus A$ by holomorphic functions. I.e., it is sufficient for our goal for every point $z_0 \in X \setminus \tilde{Y}$ and its preimages z_1, \dots, z_d to find a weakly holomorphic function $h \in \mathcal{O}'(\tilde{X})$ such that $h(z_1) = 1$ and $h(z_2) = \dots = h(z_d) = 0$. Indeed, after that our problem will be reduced to the extension of an appropriate symmetric polynomials with holomorphic coefficients from $X \setminus A$ to X , and this is standard. Construction of such h is exactly what [De] is about. One only needs to remark that the proof there "doesn't see" a codimension two "hole" A in the branch locus \tilde{Y} of the covering.

From that place we closely follow [De]. Denote by \tilde{Y}' the singular locus of \tilde{Y} and by \tilde{X}' the part of the covering situated over \tilde{Y}' . The proof then uses the fact that \tilde{Y}' is of codimension at least two in X . But then one can *include* A to \tilde{Y}' and follow the proof of [De] with these data.

Denote by \tilde{X}_0 the unbranched portion of \tilde{X} , i.e., the part of \tilde{X} situated over $X \setminus Y$. By $d\nu := c^*(\frac{1}{2})^n dz \wedge d\bar{z}$ denote the natural volume form on \tilde{X}_0 . As the first step one finds a holomorphic function h on \tilde{X}_0 , which separates preimages of z_0 as above and is square integrable with respect to $d\nu$ on \tilde{X}_0 , see Proposition 1.5 in [De]. Second, one takes the equation g of the branching locus of \tilde{X} and extends the product gh through $c^{-1}(\tilde{Y} \setminus (\tilde{Y}' \cup A))$ as in Lemma 1.5 from [NS]. And finally one extends the function $h' := gh$ through the points of $\tilde{X}' \setminus c^{-1}(A)$ as in Lemma 1.10 of [De]. Function h' possesses the required properties.

□

Remark 3.2. Remark that the branch locus \tilde{Y} of the extended covering is the closure of the branched locus of the initial covering, i.e., no new ramifications occur.

3.2. Proof of Theorem 3. As the first step of the proof we state the following:

Lemma 3.1. *Let X be a normal complex space and A a codimension two analytic subset of X . Let a holomorphic line bundle \mathcal{N} over X be given. Suppose that $\mathcal{N}|_{X \setminus A}$ admits a k -th root \mathcal{F} . Then \mathcal{F} extends as a k -th root of \mathcal{N} onto the whole of X .*

Proof. Denote, as usual by F and N the total spaces of \mathcal{F} and \mathcal{N} correspondingly. Projections onto the base in both cases will be denoted as π . There exists a natural k -linear holomorphic map $p : F \rightarrow N|_{X \setminus A}$ defined as

$$p : (x, v) \mapsto (x, v^{\otimes k}).$$

Mapping p is an analytic covering of order k with branch set $X \setminus A$. By Theorem 3.1 it extends to an analytic covering $\tilde{p} : \tilde{F} \rightarrow N$. The branched set of \tilde{p} is X , projection $\pi : F \rightarrow X \setminus A$ obviously extends to projection $\tilde{\pi} : \tilde{F} \rightarrow X$ as well as vector bundle operations. All this follows from normality of the covering space \tilde{F} and the fact that $\tilde{F} \setminus F$ is of codimension two.

The local triviality of $(\tilde{F}, \tilde{\pi}, X)$ one needs to prove over a point $a \in A$ only. Take some $(a, v) \in \tilde{\pi}^{-1}(A)$, $v \neq 0$ and let $(a, w) \in N$ be its image under \tilde{p} . Remark that $w \neq 0$, otherwise the fiber \tilde{F}_a would be contracted by \tilde{p} to a point and this violates the properness of \tilde{p} . Since \tilde{p} is a covering with the ramification set X we see that \tilde{p} is biholomorphic between some neighborhoods $V \ni (a, v)$ and $W \ni (a, w)$. If $\Sigma \subset W$ is a graph of a local section σ of \mathcal{N} then $\tilde{p}|_V^{-1}(\Sigma)$ will be the graph of a local section of $\tilde{\mathcal{F}}$.

Therefore \tilde{F} is a total space of a holomorphic line bundle $\tilde{\mathcal{F}}$ extending \mathcal{F} . It is a k -th root of \mathcal{N} because \tilde{p} was extended to.

□

Proof of Theorem 3. Let $C : X \rightarrow Y$ be the contraction. Lemma 3.1 is now applicable to the direct images of $C_*\mathcal{F}|_{X \setminus E}$ and $C_*\mathcal{N}|_{X \setminus E}$. Both are holomorphic line bundles on $Y \setminus A$. The second one extends onto Y as a coherent analytic sheaf by the Theorem about direct image sheaves of Grauert. Therefore it extends onto Y as a holomorphic line bundle. Denote this extension (with some abuse of notation) as $C_*\mathcal{N}$. By Lemma 3.1 $C_*\mathcal{F}$ extends as a holomorphic line bundle onto Y and stays there to be a k -th root of $C_*\mathcal{N}$. Denote by $C_*\mathcal{F}$ this extension. Now $C^*C_*\mathcal{F}$ will be an extension of \mathcal{F} to X , which is a k -th root of $C^*C_*\mathcal{N}$, and the last is an extension of $\mathcal{N}|_{X \setminus E}$ onto X . Theorem is proved.

□

4. LOGARITHMS OF LINE BUNDLES AND PSEUDOCONCAVE DOMAINS.

4.1. Extension of logarithms. Results of this paper in some sense can be reformulated also as extension of logarithms of holomorphic line bundles. A *logarithm* of a holomorphic line bundle $\mathcal{N} \in H^1(U, \mathcal{O}^*)$ is a cohomology class $\mathcal{H} \in H^1(U, \mathcal{O})$ such that $\exp(\mathcal{H}) = \mathcal{N}$. Here \exp stands for the map from the exponential sequence. From this sequence it is clear that \mathcal{N} admits a logarithm on U if and only if $c_1(\mathcal{N}) = 0$, i.e., if and only if \mathcal{N} is topologically trivial on U . This shows that existence of $\log \mathcal{N}$ is equivalent to the existence of all roots from \mathcal{N} .

Indeed, consider the exact sequence

$$0 \rightarrow \text{Pic}^0(U) \rightarrow \text{Pic}(U) \xrightarrow{c_1} NS(U) \rightarrow 0. \quad (4.1)$$

The Néron-Severi group $NS(U)$ is a discrete subgroup of $H^2(U, \mathbb{Z})$, in fact it is equal to $H^{1,1}(U, \mathbb{Z}) := H^2(U, \mathbb{Z}) \cap H^{1,1}(U, \mathbb{R})$, see [Che]. Therefore, would $c_1(\mathcal{N}) \neq 0$ then \mathcal{N} couldn't admit roots of infinite number of degrees. Therefore $c_1(\mathcal{N})$ should be zero and then $\mathcal{N} \in \text{Pic}^0(U)$, i.e., is topologically trivial. And vice versa, if $\log \mathcal{N}$ exists then $\mathcal{F}_k := \exp(1/k \cdot \log \mathcal{N})$ will be a k -th root of \mathcal{N} .

Corollary 4.1. *In the notations of Theorems 1 and 2 let \mathcal{H} be a logarithm of \mathcal{N} on $U_{t_0}^+$.*

i) If $n \geq 3$, or $n = 2$ and $\text{ind}\rho|_{\mathfrak{c}} \neq 2$ then \mathcal{H} extends as a cohomology class to a neighborhood of Σ_{t_0} and stays there to be a logarithm of \mathcal{N} .

ii) If $n = \text{ind}\rho|_{\mathfrak{c}} = 2$ and \mathcal{H} doesn't extend to a neighborhood of \mathfrak{c} then there exists a neighborhood V of \mathfrak{c} biholomorphic to K^2 such that $\mathcal{H}|_{K^2 \cap U_{t_0}^+}$ extends to D and as element of $H^1(D, \mathcal{O})$ is equal to $j(rA)$ for some $r \in \mathbb{Z}$, $r \neq 0$.

Proof. We use here notations from Introduction. What concerns (i) all what is needed to be remarked is that all neighborhoods to which we extended roots of \mathcal{N} do not depend on a degree of a root. This implies the topological triviality of $\mathcal{N}|_{U_t^+}$ for some $t < t_0$ close enough to t_0 . Let \mathcal{H}_1 be some logarithm of \mathcal{N} on $\mathcal{N}|_{U_t^+}$. Then $\mathcal{H}_1|_{U_{t_0}^+} = \mathcal{H} + j(\mathbf{z})$ for some $\mathbf{z} \in H^1(U_{t_0}^+, \mathbb{Z})$. But along the proof we saw that in this case the natural map $H_1(U_{t_0}^+, \mathbb{Z}) \rightarrow H_1(U_t^+, \mathbb{Z})$ is surjective. Therefore every cohomology class $\mathbf{z} \in H^1(U_{t_0}^+, \mathbb{Z})$ is a restriction of some cohomology class $\tilde{\mathbf{z}} \in H^1(U_t^+, \mathbb{Z})$. All what is left is to correct \mathcal{H}_1 by subtracting from it $j(\tilde{\mathbf{z}})$.

Consider the case (ii). Let V be a neighborhood of $\mathfrak{c} = 0$ biholomorphic to K^2 . \mathcal{H} extends to a logarithm of \mathcal{N} on $U_{t_0}^+ \cup D$ by part (i). Since $\mathcal{N}|_D$ is trivial we see from (2.8) that $\mathcal{H}|_D \in j(H^1(D, \mathbb{Z}))$, i.e., $\mathcal{H}|_D = j(rA)$ for some $r \in \mathbb{Z}$. Suppose $r = 0$. Let ω be a $(0, 1)$ -form in $U_{t_0}^+ \cup D$ representing \mathcal{H} via Dolbeault isomorphism. Since $\mathcal{H}|_D = 0$ we have that $\omega|_D = \bar{\partial}h$ for some function h in D . Take a test function φ with support in K^2 equal to 1 on $\frac{1}{2}K^2$. Then $\omega_1 := \omega - \bar{\partial}(\varphi h)$ will still represent \mathcal{H} in $H^{0,1}(U_{t_0}^+ \cup D)$. But this form is smooth on $H^{0,1}(U_{t_0}^+ \cup \frac{1}{2}K^2)$. This means that \mathcal{H} is extended to a neighborhood of \mathfrak{c} . Corollary is proved. \square

4.2. Non-extendability of roots from pseudoconcave domains. In this subsection we shall give the example of Nemirovski. Recall that a domain U in a complex manifold X is called pseudoconcave in the sense of Andreotti or, simply *pseudoconcave* if U admits a proper exhaustion function $\rho : U \rightarrow [r_0, +\infty)$ which is strictly plurisuperharmonic on $U_{r_1}^+ := \{\rho(x) > r_1\}$ for some $r_1 \geq r_0$. A typical example of such U is a complement to a Stein compact in a compact complex manifold.

Example 4.1. Let S be an imbedded surface in \mathbb{P}^2 with a basis of Stein neighborhoods and such that it is homological to a complex line. Let D be a Stein neighborhood of S .

Remark 4.1. Such S can be obtained by adding to \mathbb{P}^1 three handles and then perturbing in order to cancel all elliptic points, see [Fo] or [Ne] for more details. Genus (at least) three comes from the adjunction inequality:

$$S^2 + |c_1 \cdot S| \leq 2g - 2,$$

which is necessary and sufficient condition for an imbedded surface to have a Stein neighborhood, unless S is a sphere homological to zero.

Then the restriction of $\mathcal{O}(1)$ to $\mathbb{P}^2 \setminus D$ is topologically trivial. Indeed, write

$$H_c^2(D, \mathbb{Z}) \rightarrow H^2(\mathbb{P}^2, \mathbb{Z}) \rightarrow H^2(\mathbb{P}^2 \setminus D, \mathbb{Z}).$$

The first map is a surjection, because the class dual to S goes to the generator of $H^2(\mathbb{P}^2, \mathbb{Z})$. Therefore $H^2(\mathbb{P}^2, \mathbb{Z})$ is mapped to zero by the second arrow. In particular $c_1(\mathcal{O}(1)|_{\mathbb{P}^2 \setminus D}) =$

0. Therefore $\mathcal{O}(1)|_{\mathbb{P}^2 \setminus \bar{D}}$ admits all roots and a logarithm. But neither of them can be extended to \mathbb{P}^2 .

5. APPENDIX. HARTOGS DOMAIN IN A NEIGHBORHOOD OF A CRITICAL POINT.

We shall explain in this Appendix how to place a Hartogs figure to a upper level set of a strictly plurisubharmonic function in a neighborhood of a critical point of index two in such a way that the associated bidisc is imbedded, contains a neighborhood of this critical point and its intersection with the upper level set is connected. After that in the Remark 5.1 we shall explain once more what happens with "quasi-analytic" objects in this case, like roots of holomorphic line bundles, which satisfy (P2) but fail to satisfy (P1). We use the notations of Subsection 2.7.

Lemma 5.1. *Y_0 contains an imbedded Hartogs figure $i(H_\varepsilon^2)$ such that:*

- i) *the associated bidisc $i(\Delta^2)$ is also imbedded;*
- ii) *$i(\Delta^2) \cap Y_0$ is connected;*
- iii) *the union $i(\Delta^2) \cup Y_0$ contains a neighborhood of zero.*

Proof. Let A be the constant from (2.13), set $t_A = \frac{A^2-1}{7}$. Without loss of generality we may suppose in what follows that A is close to 1, say $1 < A < 2$. Consider the following 1-parameter family of complex annuli

$$R_{t,0} := \{z \in K^2 : (z_1 + t)^2 + z_2^2 = t_A^2\}, \quad (5.1)$$

parameterized by a real parameter $0 \leq t \leq 2t_A$. It is more convenient to work with the real form of (5.1):

$$\begin{cases} (x_1 + t)^2 + x_2^2 = \|y\|^2 + t_A^2, \\ (x_1 + t)y_1 + x_2y_2 = 0. \end{cases} \quad (5.2)$$

Let us list the needed properties of these annuli.

1_R. For every $t \in [0, 2t_A]$ the curve $R_{t,0}$ intersects the boundary ∂K^2 of the convex brick K^2 by the side $\partial K \times \text{Int} K_A$. Really, from the first equation of (5.2) we get that for $\|x\| = 1$ one has

$$\|y\|^2 \leq 1 + |2x_1t| + t^2 - t_A^2 \leq 1 + 4t_A + 3t_A^2 \leq 1 + 7t_A < A^2,$$

because of the choice of t_A . As a result we see that:

2_R. Boundaries $\partial R_{t,0}$ are entirely contained in Y_0 for all $t \in [0, 2t_A]$. This is because $\partial K \times \text{Int} K_A \subset Y_0$.

3_R. $R_{0,0}$ is entirely contained in Y_0 . This is because in that case the same equation gives $\|y\|^2 = \|x\|^2 - t_A^2 < \|x\|^2$ for all $x + iy \in R_{0,0}$.

4_R. Finally $R_{t_A,0} \ni 0$.

From 1_R–4_R we conclude that the holomorphic family of annuli

$$R_{t,\tau} := K \cap \{(z_1 + t + i\tau)^2 + z_2^2 = t_A^2\}, \quad (5.3)$$

with $t + i\tau$ in a neighborhood of U of $[0, 2t_A]$ in \mathbb{C} , sweeps up a neighborhood of zero in \mathbb{C}^2 and inherits the same properties, provided U is thin enough.

Cut each $R_{t,\tau}$ along $\{x_2 = 0, x_1 \leq -t\}$ to get the disc $D_{t,\tau} = R_{t,\tau} \setminus \{x_2 = 0, x_1 \leq -t\}$ with boundary $\partial D_{t,\tau}$. We obtain a holomorphic one parameter family of discs $D_{t,\tau}$ each catted from the annuli $R_{t,\tau}$. $D_{t,\tau}$ are mutually disjoint because $R_{t,\tau}$ are.

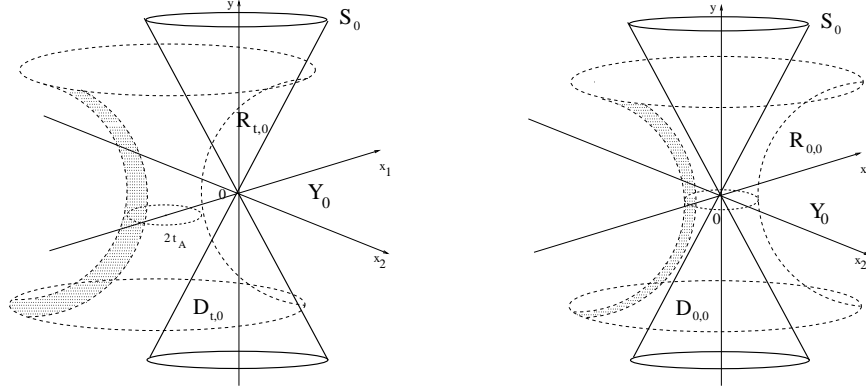


FIGURE 3. On the right the whole annulus $R_{0,0}$ is contained in Y_0 and therefore after cutting $R_{0,0}$, i.e., removing of the dashed zone on the picture, the boundary of the obtained disc $D_{0,0}$ is still in Y_0 . When $t \nearrow 2t_A$ the disc $D_{t,0}$ moves (to the left) sweeping up a neighborhood of zero, but $\partial D_{t,0}$ stays in Y_0 .

We have the following properties of discs $D_{t,\tau}$:

- 1_D. Discs $D_{t,\tau}$ fill in a domain $0 \in D := \bigcup_{t+i\tau \in U} D_{t,\tau}$ diffeomorphic to a bidisc.
- 2_D. For $t+i\tau$ close to zero $D_{t,\tau}$ is entirely contained in Y_0 .
- 3_D. Boundaries $\partial D_{t,\tau}$ are contained in Y_0 .

Really, the intersection $R_{t,\tau} \cap \{x_2 = 0, x_1 \leq -t\}$ is given by the equations $(x_1 + t)^2 = \|y\|^2 + t_A^2$ and $(x_1 + t)y_1 = 0$. Since $x_1 = -t$ is impossible we conclude that $y_1 = 0$ and get

$$\begin{cases} y_1 = x_2 = 0 \\ (x_1 + t)^2 = y_2^2 + t_A^2. \end{cases} \quad (5.4)$$

Since x_1 is negative we get immediately that $y_2^2 < x_1^2$, i.e., that $R_{t,\tau} \cap \{x_2 = 0, x_1 \leq t\} \subset Y_0$.

In what follows we cut $R_{t,\tau}$, from our family, by $\{|x_2| \leq \varepsilon, x_1 \leq -t\}$ for $\varepsilon > 0$ small enough to keep all properties 1_D – 3_D. The resulting discs we still denote as $D_{t,\tau}$ and Y stands for $\bigcup_{t+i\tau} D_{t,\tau}$ as before. Let $V \subset U$ be a sufficiently small disc centered at zero – such that for every $t+i\tau \in V$ the disc $D_{t,\tau}$ is entirely contained in Y_0 . Set

$$H_0 := \bigcup_{t+i\tau \in U} \partial D_{t,\tau} \cup \bigcup_{t+i\tau \in \frac{1}{2}\bar{V}} D_{t,\tau}. \quad (5.5)$$

By H_ε denote the ε -neighborhood of H_0 for some small $\varepsilon > 0$.

- 4_D. The intersection $Y \cap Y_0$ is connected and is contained in the envelope of holomorphy of H .

Since the boundaries of $D_{t,\tau}$ stay in Y_0 and $D_{t,\tau} \subset Y_0$ for $t+i\tau \in V$ the only thing to prove here is the connectivity of $D_{t,\tau} \cap Y_0$ for every $t+i\tau \in U$. Suppose not. Let Y_1 and Y_2 be two connected components of some $D_{t,\tau} \cap Y_0$. Both of them cannot touch the boundary of this disc, because it is connected and its neighborhood is contained in Y_0 . Therefore, one of them, say Y_1 is relatively compact in $D_{t,\tau}$. The second one could be supposed to contain the neighborhood of $\partial D_{t,\tau}$. Add to Y_2 the catted of band and obtain that $R_{t,\tau} \cap Y_0$ is not connected. If we set $X_0 := K^2 \setminus Y_0$ then that means that the compact $R_{t,\tau} \cap X_0$ has bounded components in its complement in the Riemann surface $R_{t,\tau}$.

The domain $Z_0 := X_0 \cap \text{int}(K^2)$ is the domain of holomorphy and its closure $K^2 \setminus Y_0$ is polynomially convex. Really, one can easily see that curves $\{R_{t,\tau} : t > 0\}$ pass through any given point in Y_0 , do not intersect \bar{X}_0 and leave K^2 when $t \nearrow \infty$. Therefore the polynomial convexity of $K^2 \setminus Y_0$ follows from the Oka-Stolzenberg theorem, see [St]. Intersection of a globally defined complex curve in \mathbb{C}^2 with the polynomially convex domain cannot have bounded components in its complement. This is an immediate consequence of the maximum principle.

Shrinking discs $D_{t,0}$ slightly near the boundary (and keeping $1_D - 3_D$) we can suppose that $D_{t,0}$ is a real analytic family of closed analytic discs, i.e., that there exists a real analytic map $i : [0, 2t_A] \times \bar{\Delta} \rightarrow Y$ such that i holomorphic on second variable and is a biholomorphism of $\{t\} \times \bar{\Delta}$ onto $\bar{D}_{t,0}$. This statement is nothing but the Riemann mapping theorem with parameters. The extension of i to a neighborhood of $[0, 2t_A] \times \bar{\Delta}$ will be the desired map.

□

Corollary 5.1. *Let 0 be a Morse critical point of a strictly plurisubharmonic function ρ in a neighborhood U of the origin in \mathbb{C}^n . Set $U^+ := \{z \in U : \rho(z) > 0\}$. Then U^+ contains an imbedded Hartogs figure $i(E_\varepsilon^n)$ such that:*

- i) *the associated polydisc $i(\Delta^n)$ is also imbedded and contains the origin;*
- ii) *$i(\Delta^n \setminus E_\varepsilon^n) \cap U^+$ is connected.*

The proof of this Corollary immediately follows from Lemma 5.1 after multiplication by sufficiently small Δ_ε^{n-2} .

Remark 5.1. If one deals with an analytic object one can apply condition (P1) and extend it to the constructed Hartogs figure $i(E_\varepsilon^n)$. This will extend it to the neighborhood of the critical point of ρ .

Now let us analyze what happens when one tries to extend an object σ in this situation (say a root of a globally existing bundle) which satisfies (P2) but not (P1). Then $\sigma|_{i(H_\varepsilon^2)}$ extends to σ_1 on $i(\Delta^2)$. Now we have two objects: σ on Y_0 and σ_2 on $i(\Delta^2)$. They do coincide on $i(H_\varepsilon^2)$, but why should they coincide on $i(\Delta^2) \cap Y_0$? And this is required in order to get an extension of σ to a neighborhood of zero. I.e., the very same problem that was indicated in item (c) of the Remark 2.2 happens.

Remark 5.2. (a) The proof of Lemma 5.1 is inspired by constructions from [Iv1], where a qualitative version of Bochner's tube theorem was proved, [Bo].

(b) Another way to prove this lemma is to modify the strictly plurisubharmonic function ρ in a neighborhood of its critical point, as we deed in the proof, i.e., in such a way that the difference $\bar{Y}_0 \setminus Y_0$ is a totally real disc. Then Lemma 2.20 from [Si] will do the job.

(c) In both approaches it is crucial to “catch” the critical point by **one** polydisc (with the corresponding Hartogs figure sitting in Y_0). Really, one could try to extend the bundle directly, as one can extend holomorphic functions in this situation, see [Iv1] for more details. But then one deals with the following situation: the “envelope” is $\hat{H} = \Delta \times A_{1/2,1}$ and the “Hartogs figure” is

$$H = (\Delta_\varepsilon \times A_{1/2,1}) \cup (\Delta \times V), \quad (5.6)$$

where $V := (\partial A_{1/2,1})^\varepsilon$ is an ε -neighborhood of the boundary of the annulus $A_{1/2,1}$ in \mathbb{C} , i.e., V is a union of two annuli V_1, V_2 . And now the proof of the crucial Hartogs-type

Lemma 2.1 will fail: the domain $U_1 = \Delta_\varepsilon \times A_{1/2,1}$ in this case is not simply connected and one cannot take roots from F_1 - we are using the notations of the proof of that lemma. To my best understanding it is Hartogs figures of the type (5.6) which appear in Lemma 8.1.1 of [ST].

REFERENCES

- [Bo] BOCHNER S. : *A theorem on analytic continuation of functions in several variables.* Ann. of Math. (2) **39**, no. 1, 14–19 (1938).
- [Che] CHERN S.S.: *complex manifolds.* University of Chicago (1956).
- [De] DETHLOFF G.: *A new proof of a theorem of Grauert and Remmert by L^2 -methods.* Math. Ann. **286**, 129-142 (1990).
- [Fo] FORSTNERIC F.: *Stein domains in complex surfaces.* J. Geom. Anal. **13**, no. 1, 77-94 (2003).
- [Fr] FRIEDMAN R.: *Algebraic Surfaces and Holomorphic Vector Bundles.* Universitext, Springer (1998).
- [GR1] GRAUERT H., REMMERT R.: *Theorie der Steinschen Räume.* Siperinger (1977).
- [GR2] GUNNING R., ROSSI H.: *Analytic Functions of Several complex Variables.* Prentice-Hall, N.J. (1965).
- [Iv1] IVASHKOVICH S.: *Envelopes of holomorphy of some tube sets in \mathbb{C}^2 and the monodromy theorem.* Math. USSR Izvestija **19**, N 1, 189-196 (1982).
- [Iv2] IVASHKOVICH S.: *The Hartogs type extension theorem for meromorphic maps into compact Kähler manifolds.* Invent. math. V. **109** 47-54 (1992).
- [Iv3] IVASHKOVICH S.: *Extension of analytic objects by the method of Cartan-Thullen.* In Proc. Conf. Compl. Anal. and Math. Phys. Krasnoyarsk pp.53-60 (1988).
- [Ne] NEMIROVSKI S.: *Holomorphic functions and embedded real surfaces.* Mathematical Notes **63**, N 3-4, 527-532 (1998).
- [NS] NORQUET F., SIU Y.-T.: *Holomorphic convexity of spaces of analytic cycles.* Bull. Soc. Math. France **105**, 191-223 (1977).
- [Oh] OHSAWA T.: *Nonexistence of real analytic Levi flat hypersurfaces in \mathbb{P}^2 .* Nagoya Math. J. **158**, 95-98 (2000).
- [Ro] ROYDEN H.: *The extension of regular holomorphic maps.* Proc. AMS **43**, no. 2, 306-310 (1974).
- [Si] SIU Y.-T.: *Techniques of extension of analytic objects.* Marcel Dekker, NY (1974).
- [St] STOLZENBERG G.: *Polynomially and rationally convex sets.* Acta Math. **109** 259–289 (1963).
- [ST] SIU Y.-T., TRAUTMANN G.: *Gap-Sheaves and Extension of Coherent Analytic Subsheaves.* Lect. Notes. Math **172** Springer (1971).

UNIVERSITÉ DE LILLE-1, UFR DE MATHÉMATIQUES, 59655 VILLENEUVE D’ASCQ, FRANCE

E-mail address: ivachkov@math.univ-lille1.fr

IAPMM NAT. ACAD. SCI. UKRAINE LVIV, NAUKOVA 3B, 79601 UKRAINE